ON MEAN-SQUARE BOUNDEDNESS OF STOCHASTIC LINEAR SYSTEMS WITH QUANTIZED OBSERVATIONS

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Abstract. We propose a procedure to design a state-quantizer with *finite* alphabet for a marginally stable stochastic linear system evolving in $\mathbb{R}^d$, and a bounded policy based on the resulting quantized state measurements to ensure bounded second moment in closed-loop.

1. Introduction and Result

Consider the linear control system

$$(*) \qquad x_{t+1} = Ax_t + Bu_t + w_t, \quad x_0 \text{ given}, \quad t = 0, 1, \ldots,$$

where the state $x_t \in \mathbb{R}^d$ and the control $u_t \in \mathbb{R}^m$, $(w_t)_{t\in\mathbb{N}_0}$ is a mean-zero sequence of noise vectors, and A and B being matrices of appropriate dimensions. It is assumed that instead of perfect measurements of the state, quantized state measurements are available by means of a quantizer $\mathfrak{q} : \mathbb{R}^d \longrightarrow Q$, with $Q \subset \mathbb{R}^d$ being a set of vectors in $\mathbb{R}^d$ called alphabets/bins.

Our objective is to construct a quantizer with finite alphabet and a corresponding control policy such that the magnitude of the control is *uniformly bounded*, i.e., for some $U_{\max} > 0$ we have $\|u_t\| \leqslant U_{\max}$ for all t, the number of alphabets Q is *finite*, and the states of $(*)$ are *mean-square bounded* in closed-loop.

Stabilization with quantized state measurements has a rich history, see e.g., [2, 4, 1, 3, 7, 8] and the references therein. While most of the literature investigates quantization techniques for stabilization under communication constraints, especially of systems with eigenvalues outside the closed unit disc, our result is directed towards "maximally coarse" quantization—with finite alphabet of Lyapunov stable systems; communication constraints are not addressed in this work. The authors are not aware of any prior work dealing with stabilization with finite alphabet in the context of Lyapunov stable systems. Observe that unlike deterministic systems, local stabilization of stochastic systems with unbounded noise, at least one eigenvalue with magnitude greater than 1, and bounded inputs is impossible.

Assumption 1.

- The matrix A is Lyapunov stable—the eigenvalues of A have magnitude at most 1, and those on the unit circle have equal geometric and algebraic multiplicities.
- The pair (A, B) is reachable in κ steps, i.e., $\operatorname{rank}\begin{pmatrix} B & AB & \cdots & A^{\kappa-1}B \end{pmatrix} = d$.
- $(w_t)_{t\in\mathbb{N}_0}$ is a mean-zero sequence of mutually independent noise vectors satisfying $C_4 := \sup_{t\in\mathbb{N}_0} \mathsf{E}[\|w_t\|^4] < \infty$.
- $\|u_t\| \leqslant U_{\max}$ for all $t \in \mathbb{N}_0$. $\diamond$

The policy that we construct below belongs to the class of κ-history-dependent policies, where the history is that of the quantized states. We refer the reader to our earlier article [6] for the basic setup, various definitions, and in particular to [6, §3.4] for the details about a change of basis in $\mathbb{R}^d$ that shows that it is sufficient to consider A orthogonal. We let $\mathcal{R}_k(A, M) := \begin{pmatrix} A^{k-1}M & \cdots & AM & M \end{pmatrix}$ for a matrix M of appropriate dimension, $M^+ := M^\mathsf{T}(MM^\mathsf{T})^{-1}$ denote the Moore-Penrose pseudoinverse of M in case the latter has full row rank, and $\sigma_{\min}(M), \sigma_{\max}(M)$ denote the minimal and maximal singular values of

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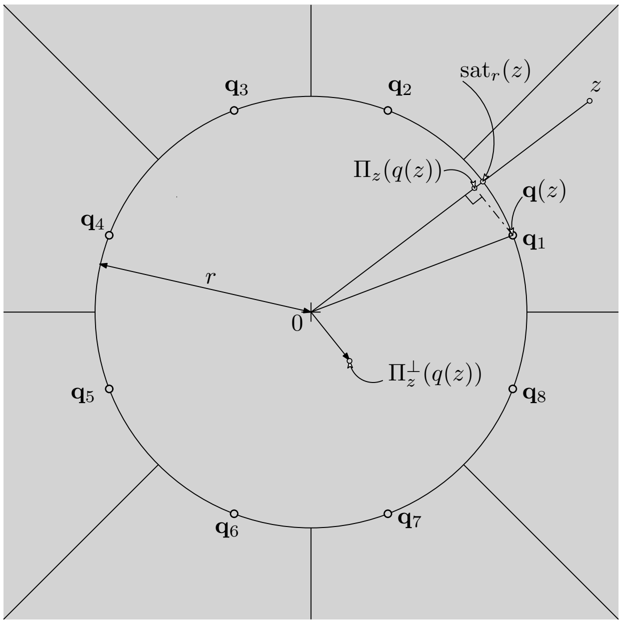


FIGURE 1. Pictorial depiction of the proposed quantization scheme in $\mathbb{R}^2$, with $\{\mathbf{q}_0 = 0, \mathbf{q}_1, \ldots, \mathbf{q}_8\}$ being the set of bins. The various projections are computed for a generic state z outside the r-ball centered at the origin.

M, resp. I denotes the $d \times d$ identity matrix. For a vector $v \in \mathbb{R}^d$, let $\Pi_v(\cdot) := \langle \cdot, \frac{v}{\|v\|}\rangle \frac{v}{\|v\|}$ and $\Pi_v^\perp(\cdot) := I - \Pi_v(\cdot)$ denote the projections onto the span of v and its orthogonal complement, respectively. For $r > 0$ the radial r-saturation function $\mathrm{sat}_r : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ defined as $\mathrm{sat}_r(y) := \min\{r, \|y\|\} \frac{y}{\|y\|}$, and let $B_r \subset \mathbb{R}^d$ denote the open r ball centered at 0 with ∂B_r being its boundary. We have the following theorem:

Theorem 2. *Consider the system (*), and suppose that Assumption 1 holds. Assume that the quantizer is such that there exists a constant r satisfying:*

a) $r > \dfrac{\sqrt{\kappa}\,\sigma_{\max}(\mathcal{R}_\kappa(A, I))\sqrt[4]{C_4}}{\cos(\varphi) - \sin(\varphi)}$, *where* $\varphi \in [0, \pi/4[$ *is the maximal angle between* z *and* $\mathfrak{q}(z)$, $z \notin B_r$, *and*

b) $\mathfrak{q}(z) = \mathfrak{q}(\mathrm{sat}_r(z)) \in \partial B_r$ *for every* $z \notin B_r$.

Finally assume that that $U_{\max} \geqslant r/\sigma_{\min}(\mathcal{R}_\kappa(A, B))$. *Then successive* κ*-step applications of the control policy*

$$\begin{pmatrix} u_{\kappa t} \\ \vdots \\ u_{\kappa(t+1)-1} \end{pmatrix} := -\mathcal{R}_\kappa(A, B)^+ A^\kappa \mathfrak{q}(x_{\kappa t}), \qquad t \in \mathbb{N}_0,$$

ensures that $\sup_{t \in \mathbb{N}_0} \mathsf{E}_{x_0}[\|x_t\|^2] < \infty$.

Observe that Theorem 2 outlines a procedure for constructing a quantizer with finitely many bins, an example of which on $\mathbb{R}^2$ is depicted in Figure 1. We see from the hypotheses of Theorem 2 that the quantizer has no large gap between the bins on the r-sphere, and is "radial"; the quantization rule for states inside B_r does not matter insofar as mean-square boundedness of the states is concerned. As a consequence of the control policy in Theorem 2, the control alphabet is also finite with $\kappa\,|Q|$ elements. Moreover, note that as $\varphi \searrow 0$, i.e., as the "density" of the bins on the r-sphere increases, we recover the policy proposed in [6], and in particular, the lower bound on $U_{\max}$ in [6].

2. Proof of Theorem 2

We assume that the random variables w_t are defined on some probability space $(\Omega, \mathfrak{F}, \mathsf{P})$. Herereafter $\mathsf{E}^{\mathfrak{F}'}[\cdot]$ denotes conditional expectation for a σ-algebra $\mathfrak{F}' \subset \mathfrak{F}$.

We need the following immediate consequence of [5, Theorem 1].

Proposition 3. *Let $(\xi_t)_{t\in\mathbb{N}_0}$ be a sequence of nonnegative random variables on some probability space $(\Omega, \mathfrak{F}, \mathsf{P})$, and let $(\mathfrak{F}_t)_{t\in\mathbb{N}_0}$ be any filtration to which $(\xi_t)_{t\in\mathbb{N}_0}$ is adapted. Suppose that there exist constants $b > 0$, and $J, M < \infty$, such that $\xi_0 \leqslant J$, and for all t:*

$$\mathsf{E}^{\mathfrak{F}_t}[\xi_{t+1} - \xi_t] \leqslant -b \quad \text{on the event } \{\xi_t > J\}, \quad \text{and}$$

$$\mathsf{E}[|\xi_{t+1} - \xi_t|^4 \,|\xi_0, \ldots, \xi_t] \leqslant M.$$

Then there exists a constant $\gamma = \gamma(b, J, M) > 0$ such that $\sup\limits_{t\in\mathbb{N}_0} \mathsf{E}[\xi_t^2] \leqslant \gamma$.

Proof of Theorem 2: Let $\mathfrak{F}_t$ be the σ-algebra generated by $\{x_s \mid s = 0, \ldots, t\}$. Since $\mathfrak{q}$ is a measurable map, it is clear that $(\mathfrak{q}(x_t))_{t\in\mathbb{N}_0}$ is $(\mathfrak{F}_t)_{t\in\mathbb{N}_0}$-adapted.

We have, for $t \in \mathbb{N}_0$, on $\{\|x_{\kappa t}\| > r\}$,

$$\mathsf{E}^{\mathfrak{F}_{\kappa t}}\left[\left\|x_{\kappa(t+1)}\right\| - \|x_{\kappa t}\|\right] = \mathsf{E}^{\mathfrak{F}_{\kappa t}}\left[\|A^\kappa x_{\kappa t} + \mathcal{R}_\kappa(A,B)\bar{u}_{\kappa t} + \bar{w}_{\kappa t}\| - \|x_{\kappa t}\|\right],$$

where $\bar{u}_{\kappa t} := \begin{pmatrix} u_{\kappa t} \\ \vdots \\ u^{\top}_{\kappa(t+1)-1}\end{pmatrix} \in \mathbb{R}^{\kappa m}$, and $\bar{w}_{\kappa t} := \mathcal{R}_\kappa(A, I)\begin{pmatrix} w_{\kappa t} \\ \vdots \\ w_{\kappa(t+1)-1}\end{pmatrix} \in \mathbb{R}^{\kappa d}$ is zero-mean noise. To wit, we have

$$\begin{aligned}
&\mathsf{E}^{\mathfrak{F}_{\kappa t}}\left[\|A^\kappa x_{\kappa t} + \mathcal{R}_\kappa(A,B)\bar{u}_{\kappa t} + \bar{w}_{\kappa t}\| - \|x_{\kappa t}\|\right] \\
&\qquad \leqslant \mathsf{E}^{\mathfrak{F}_{\kappa t}}\left[\|A^\kappa x_{\kappa t} + \mathcal{R}_\kappa(A,B)\bar{u}_{\kappa t}\| - \|x_{\kappa t}\|\right] + \sqrt{\kappa}\,\sigma_{\max}(\mathcal{R}_\kappa(A,I))\sqrt[4]{C_4}.
\end{aligned}$$

Selecting the controls $\bar{u}_{\kappa t} = -\mathcal{R}_\kappa(A,B)^+ A^\kappa \mathfrak{q}(x_{\kappa t})$ as in the theorem and using the fact that $\mathfrak{q}(x_{\kappa t}) = \Pi_{x_{\kappa t}}(x_{\kappa t}) + \Pi^\perp_{x_{\kappa t}}(x_{\kappa t})$, we arrive at

$$\begin{aligned}
&\mathsf{E}^{\mathfrak{F}_{\kappa t}}\left[\|A^\kappa x_{\kappa t} + \mathcal{R}_\kappa(A,B)\bar{u}_{\kappa t}\| - \|x_{\kappa t}\|\right] \\
&\quad = \|A^\kappa x_{\kappa t} - A^\kappa \mathfrak{q}(x_{\kappa t})\| - \|x_{\kappa t}\| \\
&\quad \leqslant \|A^\kappa x_{\kappa t} - \mathrm{sat}_r(A^\kappa x_{\kappa t})\| - \|x_{\kappa t}\| + \left\|\Pi^\perp_{x_{\kappa t}}(A^\kappa \mathfrak{q}(x_{\kappa t}))\right\| \\
&\qquad + \left\|\mathrm{sat}_r(A^\kappa x_{\kappa t}) - \Pi_{x_{\kappa t}}(A^\kappa \mathfrak{q}(x_{\kappa t}))\right\| \\
&\quad = -r + \left\|A^\kappa\, \mathrm{sat}_r(x_{\kappa t}) - \Pi_{x_{\kappa t}}(A^\kappa \mathfrak{q}(x_{\kappa t}))\right\| + \left\|\Pi^\perp_{x_{\kappa t}}(A^\kappa \mathfrak{q}(x_{\kappa t}))\right\| \\
&\quad = -r + \left\|A^\kappa\, \mathrm{sat}_r(x_{\kappa t}) - \Pi_{x_{\kappa t}}(A^\kappa \mathfrak{q}(\mathrm{sat}_r(x_{\kappa t})))\right\| \\
&\qquad + \left\|\Pi^\perp_{x_{\kappa t}}(A^\kappa \mathfrak{q}(\mathrm{sat}_r(x_{\kappa t})))\right\| \quad \text{by hypothesis b)} \\
&\quad \leqslant -r + r(1 - \cos(\varphi)) + r\sin(\varphi) \\
&\quad \leqslant -b \quad \text{for some } b > 0 \text{ by hypothesis a)}.
\end{aligned}$$

Moreover, we see that for $t \in \mathbb{N}_0$, since A is orthogonal,

$$\begin{aligned}
\mathsf{E}\left[\left|\left\|x_{\kappa(t+1)}\right\| - \|x_{\kappa t}\|\right|^4 \,\middle|\, \{\|x_{\kappa s}\|\}_{s=0}^t\right] &= \mathsf{E}\left[\left|\left\|x_{\kappa(t+1)}\right\| - \|A^\kappa x_{\kappa t}\|\right|^4 \,\middle|\, \{\|x_{\kappa s}\|\}_{s=0}^t\right] \\
&= \mathsf{E}\left[\left|\|A^\kappa x_{\kappa t} + \mathcal{R}_\kappa(A,B)\bar{u}_{\kappa t} + \mathcal{R}_\kappa(A,I)\bar{w}_{\kappa t}\| - \|A^\kappa x_{\kappa t}\|\right|^4 \,\middle|\, \{\|x_{\kappa s}\|\}_{s=0}^t\right] \\
&\leqslant \mathsf{E}\left[\|\mathcal{R}_\kappa(A,B)\bar{u}_{\kappa t} + \mathcal{R}_\kappa(A,I)\bar{w}_{\kappa t}\|^4 \,\middle|\, \{\|x_{\kappa s}\|\}_{s=0}^t\right] \leqslant M
\end{aligned}$$

for some $M > 0$ since $\bar{u}_{\kappa t}$ is bounded in norm and $\mathsf{E}[\|w_t\|^4] \leqslant C_4$ for each t.

We let $J := \max\{\|x_0\|, r\}$. It remains to define $\xi_t := \|x_{\kappa t}\|$ and appeal to Proposition 3 with the above definition of $(\xi_t)_{t\in\mathbb{N}_0}$ to conclude that there exists some $\gamma = \gamma(b, J, M) > 0$ such that $\sup_{t\in\mathbb{N}_0} \mathsf{E}[\xi_t^2] = \sup_{t\in\mathbb{N}_0} \mathsf{E}_{x_0}[\|x_{\kappa t}\|^2] \leqslant \gamma$. A standard argument, e.g., as in [6,

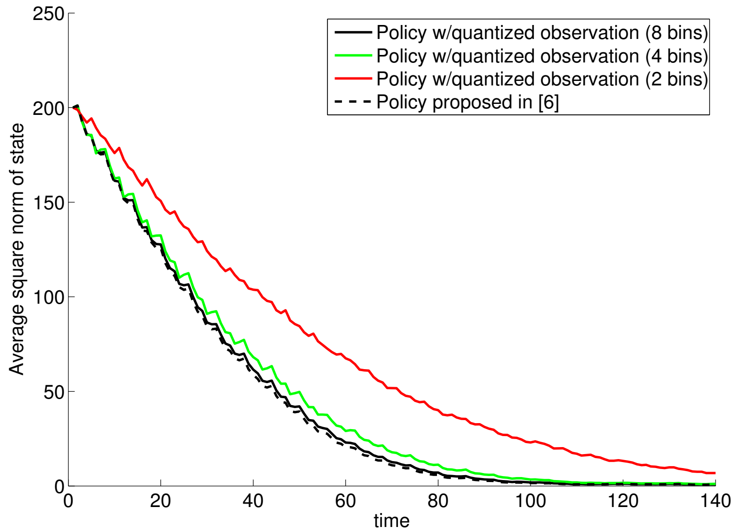
Proof of Lemma 9], shows that this is enough to guarantee $\sup_{t\in\mathbb{N}_0} \mathsf{E}_{x_0}[\|x_t\|^2] \leqslant \gamma'$ for some $\gamma' > 0$. □

3. Simulation

The figure below shows the average of the square norm of the state over 1000 runs of the system:

$$x_{t+1} = \begin{pmatrix} \cos(\pi/3) & -\sin(\pi/3) \\ \sin(\pi/3) & \cos(\pi/3) \end{pmatrix} x_t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_t + w_t,$$

where $x_0 = (10 \quad 10)^\mathsf{T}$, $w_t \in \mathcal{N}(0, I_2)$, and where u_t is chosen respectively according to the policy proposed in this article and the one proposed in [6].



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